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# THE ANTISYMMETRIC DIURNAL TIDE

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# THE ANTISYMMETRIC DIURNAL TIDE

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ABSTRACT. We propose a rapid and precise method to solve the Laplace tidal equations. Using this method, and taking into account the different duration of the solar and sideral day, we have calculated the oscillatory modes of the antisymmetric diurnal tide. We show that the ensemble of the Hough functions is complete, each mode being associated with one of these functions. The variations of tidal mode with altitude can be deduced from their equivalent depths.

## INTRODUCTION

The variation in tides with latitude is given by the Laplace equation, for which Hough [1897, 1898] gave the first method for resolution. Since this important work, these solutions have been called Hough functions. /1*

Among the most significant tides, the diurnal tide is the last to be solved — firstly, because the semidiurnal tide has the greatest amplitude at ground level, and secondly, because the equations present difficulties arising from the existence of so-called critical latitudes which do not give rise to singularities [Brilloun, 1932] which differentiate it from other tides by the presence of equivalent negative levels [Lindzen, 1966; Kato, 1966]. We are going to explain a method which allows rapid and precise resolution of the Laplace tidal equations — a method we have used to compute the modes of the antisymmetric diurnal tide. Our results show that the antisymmetric

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* Numbers in the margin indicate the pagination of the original foreign text.

(1) Aeronomy Service, C.N.R.S. 91-Verrieres-le-Buisson.

diurnal tide has modes which can be propagated upward far from the point of excitation without being reflected, as is the case for modes which have a very large or negative equivalent depth.

## I. THE METHOD FOR SOLUTION

The Laplace tidal equation is written [Siebert, 1961]:

$$\frac{d}{d\mu}\left[\frac{1-\mu^2}{f^2-\mu^2}\frac{d\Theta^s_{\lambda,n}}{d\mu}\right]-\frac{1}{f^2-\mu^2}\left[\frac{s f^2+\mu^2}{f f^2-\mu^2}+\frac{s^2}{1-\mu^2}\right]\Theta^s_{\lambda,n}+\frac{4\,a^2\omega^2}{g h_n}\Theta^s_{\lambda,n}=0. \qquad (1)$$

with $\mu = \cos\theta$ (for the rest of the notation, we refer to our work in this same journal). /2

We have here an equation with eigenvalues and eigenfunctions. The latter are Hough functions. The equivalent depth h, which is one of the factors of the eigenvalue, appears as a separation constant when one assumes in the velocity divergence equation that this depth is the product of a function of depth multiplied by a function of $\Phi,\theta$ and t [Wilkes, 1949; Siebert, 1961].

The expansion of Hough functions in one series of associated Legendre functions yields a very rapid convergence. The problem is so constructed that one may seek a solution in the following way. If we write Equation (1):

$$O_p(\mu)\,\Theta^s_{\lambda,n} = \delta^s_{\lambda,n}\,\Theta^s_{\lambda,n}$$

with

$$O_p(\mu)=\frac{d}{d\mu}\left[\frac{1-\mu^2}{f-\mu^2}\frac{d}{d\mu}\right]-\frac{1}{f^2-\mu^2}\left[\frac{s f^2+\mu^2}{f f^2-\mu^2}+\frac{s^2}{1-\mu^2}\right]\delta^s_{\lambda,n}=\frac{4a^2\omega^2}{g h_n}.$$

We would then have to compute $\Pi m,n=\int_{-1}^{1}P^s_m\,O_p(\mu)\,P^s_n\,d\mu$ to construct a symmetrical matrix which one may diagonalize to find $\delta^s_{\lambda,n}$.

Unfortunately, the $\Pi$ m,n give a divergent matrix, and the method cannot be used.

We are going to show that, using a relation given by Hough, a matrix can be constructed which (though not symmetrical) is convergent and has eigenvalues which are values of h with a known constant.

The Hough relation is:

$$\alpha_{n-2}^{s} C_{n-2}^{s} + (\beta_{n}^{s} - P) C_{n}^{s} + \gamma_{n}^{s} C_{n+2}^{s} = 0. \qquad (2)$$

$C_n^S$ is the coefficient of the Legendre associated function $P_n^S$ in the expansion of the Hough function $C_{s-2}^{s} = C_{s-1}^{s} = 0$ ,

n = s, s + 2, s + 4... for symmetric tides

= s + 1, s + 3, s + 5 for the antisymmetric tides.

$$\alpha_{n}^{s} = \frac{(n-s+1)\ (n-s+2)}{(2n+1)(2n+3)[(n+1)\ (n+2)-s/f]} \qquad (3)$$

$$\beta_{n}^{s} = -f\frac{2n(n+1)-s/f}{n^{2}(n+1)^{2}} + \frac{(n-1)^{2}(n-s)(n+s)}{n^{2}(2n+1)(2n-1)[(n-1)n-s/f]} + \frac{(n+2)^{2}(n-s+1)(n+s+1)}{(n+1)^{2}(2n+1)(2n+3)[(n+1)(n+2)-s/f]} \qquad (4)$$

$$\gamma_{n}^{s} = \frac{(n+s+1)(n+s+2)}{(2n+3)(2n+5)\ [(n+1)(n+2)-s/f]} \qquad (5)$$

$$P = -\frac{hg}{4a^{2}\omega^{2}} \qquad (6)$$

Expression (2) consitutes an infinite set of homogeneous algebraic equations in which the unknowns are $C_n^S$.

The matrix of coefficients

$$\begin{vmatrix} \beta_2^1 - P & \gamma_2^1 & 0 & 0 & \cdots \\ \alpha_2^1 & (\beta_4^1 - P) & \gamma_4^1 & 0 & \cdots \\ 0 & \alpha_4^1 & (\beta_6^1 - P) & \gamma_6^1 & \cdots \\ \cdot & \cdots & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdots & \cdot \end{vmatrix} = 0 \qquad (7)$$

is also an equation with eigenvalues. The matrix:

$$\begin{vmatrix} \beta_2^1 & \gamma_2^1 & 0 & 0 & \cdot & \cdot \\ \alpha_2^1 & \beta_4^1 & \gamma_4^1 & 0 & \cdot & \cdot \\ 0 & \alpha_4^1 & \beta_6^1 & \gamma_6^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \qquad (8)$$

is a non-symmetrical, convergent, tri-diagonal matrix. In general, one cannot diagonalize a matrix of this type, but it can be made triangular which amounts to the same thing for the purpose of obtaining the eigenvalues. It is known that the eigenvalues are real [Kato, 1966], and for that reason we have selected the Francis [1961 - 62] QR algorithmic method to attain supertriangulation. This method is applicable when the matrix is reduced to the Hessemberg form, i.e., with aij = 0 for $i > j + 1$ .

It is apparent that Matrix (8) is a simplified Hessemberg form. We are going to say a few words about the Francis method. We call $M^{(0)}$ the initial matrix, the elements of which are $a_{nm}$.

At the outset, we must make a substitution

$$M^{(0)} - \rho^{(0)} I$$

where $\rho^{(0)}$ is determined by the smallest value taken from

$$|\tau_1^{(0)} - ann| \qquad \text{and} \qquad |\tau_2^{(0)} - ann|$$

where $\tau_1^{(0)}$ and $\tau_2^{(0)}$ are the eigenvalues of 2 x 2 block lower right.

By then effecting rotation in the planes

$$R(n-1,n) \dots \quad R(1,2)\,(M^{(0)} - \rho^{(0)} I) = R^{(0)}$$

one obtains

$$M^{(1)} = R^{(0)}\,(R(1,2))^T \dots \quad (R(n-1,n))^T + \rho^{(0)} I$$

Iterations of the same type are continued.

When $M^{(s)}$ is reached in which $a_{n,n-1}$ is negligible, $a_{n,n}$ is taken as the /3 eigenvalue. The last row and column are deleted, and we continue to work with smaller and smaller matrices.

The eigenvalues are given in decreasing order of their absolute values.

We have done the calculations for the symmetrical diurnal tide with f = 1/2 in such a way as to be able to compare the results with those of Katô.

Using a matrix with 20 x 20 elements, we have obtained the same results for P that the author obtained using the continuous fraction method with 100 terms.

This work took us less than two minutes using the IBM 7040 at Meudon.

We must note that in these computations we obtained other values of P.

We retain the eigenvalues, which do not vary when the order of the truncated matrix is varied.

## II. THE ANTISYMMETRIC DIURNAL TIDE

If we do not take into account the difference between solar and sidereal days, Hough functions are all orthogonals with the associated Legendre function $P_2^1$ [Lindzen, 1965]. They are not sufficient to expand all the heating functions, since they do not constitute a complete system. If one takes account of the difference as is normally done for the other tides, f is equal to 0.498635, rather than 1/2.

In this case, Equation (1) allows antisymmetric solutions which we have computed by the preceding method. Values of h are given by the expression

$$P = 0.011349 \text{ h (km)}.$$

The table shows the values of h and the coefficients for the primary functions. Coefficients are given for normalized Hough functions and normalized associated Legendre functions. We have found equivalent negative and positive depths for the antisymmetric diurnal tide. We designate the functions $\Theta_1^1(\pm n)$ with $n = \pm (s + \nu)$ being the number of modes for the function in the interval being defined.

Figures 1, 2, 3, 4, and 5 represent the variation with latitude of functions and of NS and EW wind components. Although the equivalent depth for the first mode is positive, it is so large that the solution of the radial equation will necessarily be exponential. If we accept that among these solutions the only possible ones are those that decrease with altitude, the first mode cannot play an important role in the upper layers of the atmosphere, far from sources of excitation. The $2^{nd}$, $3^{rd}$ and $4^{th}$ modes have equivalent negative depths. The same factors apply, therefore, as for the $1^{st}$ mode.

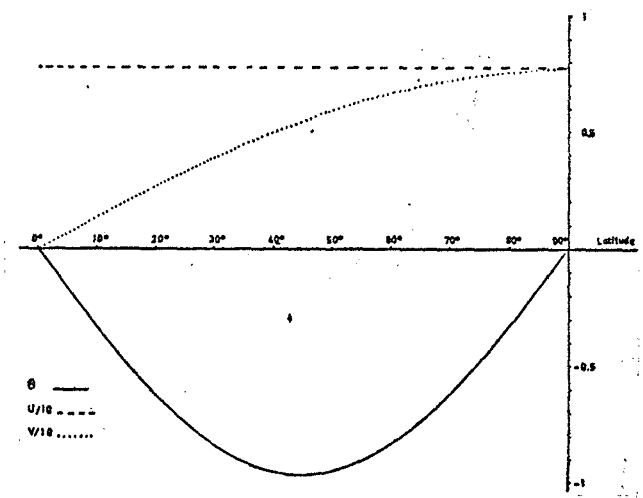


Figure 1. Variation with latitude of $\Theta^1_{1,2}$ $\nu$ and u.

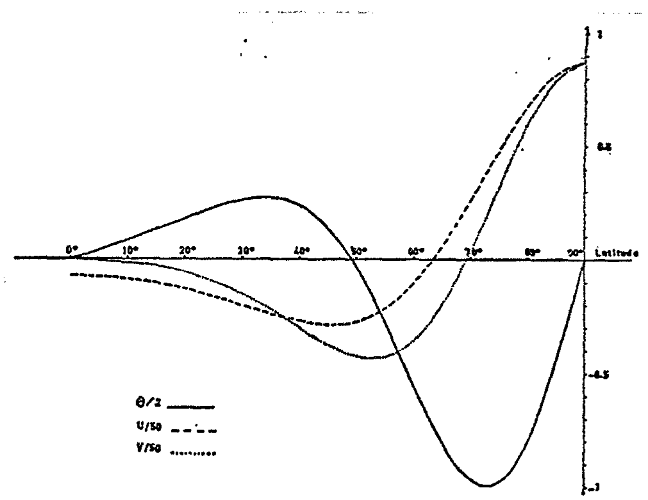


Figure 2. Variation with latitude of $\Theta^1_{1,(-4)}$ $\nu$ and u.

The $5^{th}$ mode has a positive equivalent depth with a value which must give a sinusoidal type solution to the radial equation. This mode can be propagated upward and can cross (without reflection) the negative temperature gradient zone below 80 km.

TABLE

EQUIVALENT DEPTHS AND COEFFICIENTS FOR EXPANSION OF HOUGH FUNCTIONS IN A SERIES OF ASSOCIATED LEGENDRE FUNCTIONS.

| | $\Theta^1_{1,2}$ | $\Theta^1_{1,(-4)}$ | $\Theta^1_{1,(-6)}$ | $\Theta^1_{1,(-8)}$ | $\Theta^1_{1,4}$ | $\Theta^1_{1,(-10)}$ | $\Theta^1_{1,(-12)}$ | $\Theta^1_{1,(-14)}$ | $\Theta^1_{1,6}$ |
|---|---|---|---|---|---|---|---|---|---|
| $h$ [km] | 803 | — 1,814 | — 0,646 | — 0,329 | 0,238 | — 0,199 | — 0,133 | — 0,095 | 0,072 |
| $C^1_2$ | 0,9999 | 0,0021 | 0,0009 | 0,0005 | 0,0007 | 0,0003 | 0,0002 | 0,0002 | 0,0003 |
| $C^1_4$ | - 0,0025 | 0,8199 | 0,3502 | 0,2068 | 0,2879 | 0,1407 | 0,1037 | 0,0805 | 0,1184 |
| $C^1_6$ | | 0,5512 | - 0,2983 | - 0,2615 | - 0,5774 | - 0,2018 | - 0,1577 | - 0,1264 | - 0,2118 |
| $C^1_8$ | | 0,1522 | - 0,7288 | - 0,2425 | 0,6013 | - 0,0859 | - 0,0294 | - 0,0066 | 0,1088 |
| $C^1_{10}$ | | 0,0237 | - 0,4762 | 0,4134 | - 0,4145 | 0,3685 | 0,2776 | 0,2110 | 0,1717 |
| $C^1_{12}$ | | 0,0024 | - 0,1695 | 0,6608 | 0,2079 | 0,1124 | - 0,0759 | - 0,1284 | - 0,4444 |
| $C^1_{14}$ | | 0,0002 | - 0,0394 | 0,4341 | - 0,0799 | - 0,4584 | - 0,3772 | - 0,2406 | 0,5530 |
| $C^1_{16}$ | | | - 0,0065 | 0,1765 | 0,0243 | - 0,6100 | - 0,0098 | 0,1983 | - 0,4870 |
| $C^1_{18}$ | | | - 0,0008 | 0,0506 | - 0,0060 | - 0,4061 | 0,4745 | 0,3443 | 0,3361 |
| $C^1_{20}$ | | | | 0,0109 | 0,0012 | - 0,1805 | 0,5708 | - 0,0657 | - 0,1904 |
| $C^1_{22}$ | | | | 0,0018 | - 0,0002 | - 0,0593 | 0,3858 | - 0,4777 | 0,0910 |
| $C^1_{24}$ | | | | 0,0002 | | - 0,0152 | 0,1830 | - 0,5396 | - 0,0374 |
| $C^1_{26}$ | | | | | | - 0,0031 | 0,0664 | - 0,3700 | 0,0133 |
| $C^1_{28}$ | | | | | | - 0,0005 | 0,0193 | - 0,1847 | - 0,0042 |
| $C^1_{30}$ | | | | | | | 0,0046 | - 0,0723 | 0,0011 |
| $C^1_{32}$ | | | | | | | 0,0009 | - 0,0231 | - 0,0003 |
| $C^1_{34}$ | | | | | | | 0,0001 | - 0,0062 | |
| $C^1_{36}$ | | | | | | | | - 0,0014 | |
| $C^1_{38}$ | | | | | | | | - 0,0002 | |

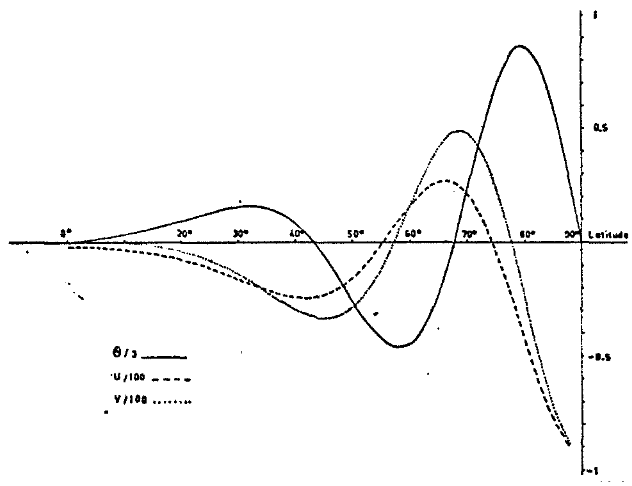


Figure 3. Variation with latitude of $\Theta^1_{1,(-6)}$ $v$ and u.

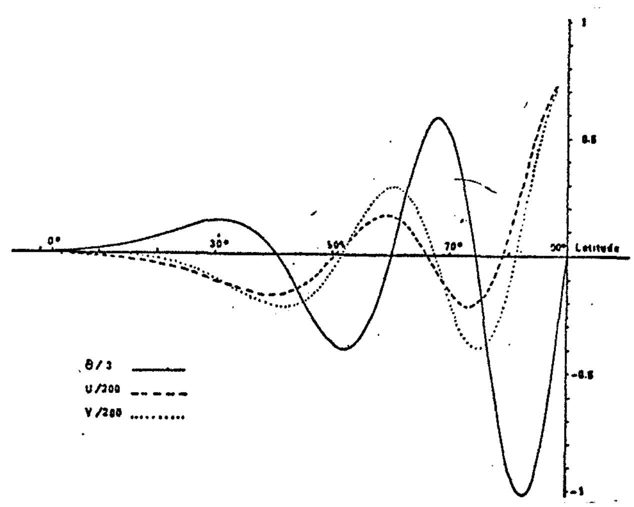


Figure 4. Variation with latitude of $\Theta^1_{1,(-8)}$ $v$ and u

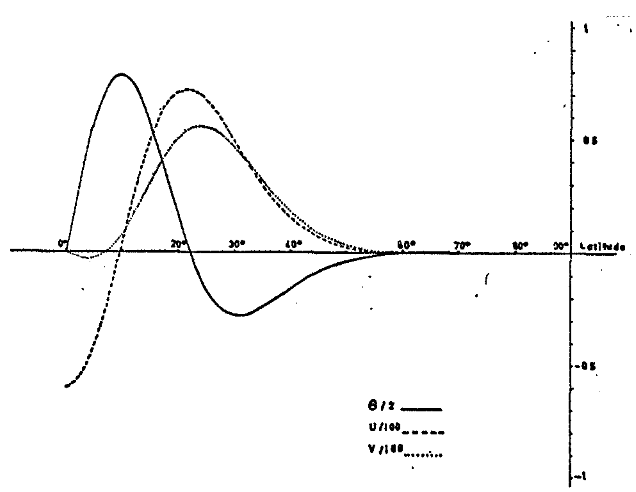


Figure 5. Variation with latitude of $\Theta^1_{1,4}$ $v$ and u.

The length of the vertical wave for such an oscillation (computed from the 1965 CIRA model) is on the order of 12 km between altitudes of 90 and 120 km.

## CONCLUSION

We have found a method for computing solutions of the Laplace tidal equations by devising an equation with eigenvalues. This method allows us to obtain equivalent tidal depths quickly and to a close approximation. In particular, we have computed the antisymmetric diurnal tide which is as easy to calculate as the others if one takes into consideration the difference between solar and sidereal days.

With the Hough functions thus complete, all excitation functions can be expanded in a series of these functions. We have found (similar to the symmetric tide) equivalent positive and negative depths, and (except for the first mode) a confinement phenomenon around the equator for the positive depth and beyond 50° latitude for the negative depth.

Among the first (and in keeping with theory) there is only $S^1_{1.4}$ which could be observed in the upper atmosphere.

If one considers carefully the seasonal variation for the diurnal tide at 93 km altitude as given by Roper [1966], it is apparent that there are two large peaks at the equinoxes and two small peaks at the soltices. The two large peaks must correspond to symmetric modes, and the two small peaks to antisymmetric modes — in particular $S^1_{1.4}$

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## REFERENCES


1. Hough, S. S. "On the Application of Harmonic Analysis to the Dynamical Theory of Tides," Phil. Trans. Roy. Soc. (London), No. A 189, 1897 - 1898, pp. 201 - 257.

2. Lindzen, R. S. "On the Theory of the Diurnal Tide" Month., Weather Re, No. 94, 1966, p. 295 - 301.

3. Kato, S. "Diurnal Atmospheric Oscillation. 1. Eigenvalues and Hough Functions," J.G.R., No. 71, 1966, pp. 3201 - 3209.

4. Brillouin, M. "Les Latitudes Critques"(The Critical Latitudes) C. R. Acad. Sciences, No. 194, 1932, pp. 801 - 804.

5. Wilkes, M. V. Oscillations of the Earth's Atmosphere, Cambridge University Press, 1949.

6. Siebert, M. Advances in Geophysics, No. 7, Academic Press, 1961, pp. 105 - 187.

7. Francis, J.G.F. "The Q.R. Transformation Part I and Part II," Computer Journal. No. 4, 1961 - 1962, pp. 265 - 271, 332 - 345.

8. Lindzen, R. S. "On the Asymmetric Diurnal Tide," Pure and App. Geoph. No. 62, 1965, p. 142.

9. Roper, R. G. "Atmospheric Turbulence in the Meteoric Region," J.G.R. No. 71, 1966, pp. 5785 - 5792.


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